Dimension of SLE curves.

$\mathrm{Hdim}\, K := \inf_{\alpha} \{ \forall \varepsilon > 0\ \exists\, (B(x_j, \delta_j)) : \sum \delta_j^{\alpha} < \varepsilon,\ \bigcup B(x_j, \delta_j) \supset K \}.$

Hausdorff dimension.

Theorem (Beffara). For $\kappa \le 8$,
$\mathrm{Hdim}(SLE_\kappa) = 1 + \frac{\kappa}{8}$ a.s.

We will prove an upper bound.

Let $N(\delta, K) := \min \# \{ B(x_j, \delta) : K \subset \bigcup B(x_j, \delta) \}$.

Upper box (Minkowski) dimension:

$$\overline{\mathrm{Mdim}}(K) := \overline{\lim} \frac{\log N(\delta, K)}{\log \frac{1}{\delta}}$$

Lower box (Minkowski) dimension:

$$\underline{\mathrm{Mdim}}(K) := \underline{\lim} \frac{\log N(\delta, K)}{\log \frac{1}{\delta}}$$

Observe: $\mathrm{Hdim}\, K \le \underline{\mathrm{Mdim}}(K) \le \overline{\mathrm{Mdim}}(K)$

Indeed, if $\alpha > \underline{\mathrm{Mdim}}(K)$ $\exists\, \delta_j \downarrow 0 : N(\delta_j, K) < \left(\frac{1}{\delta_j}\right)^{\alpha - \varepsilon} \Rightarrow$
$\sum \delta_j^{\alpha} = \delta_j^{-\alpha+\varepsilon} \delta_j^{\alpha} \to 0$ as $\delta_j \to 0$. So $\alpha \ge \mathrm{Hdim}\, K$

Lemma (an easy upper bound) Let K be a random set and $\overline{\lim} \frac{\log \mathbb{E}(N(\delta, K))}{|\log \delta|} < \alpha$ $(\alpha > 0)$. Then $\overline{\mathrm{Mdim}}\, K \le \alpha$

Pf. Take $\alpha_1 > \alpha_2 > \alpha$. Then, for small δ,
$E(N(K,\delta)) \le \delta^{-\alpha_2}$. so $\mathbb{P}(N(K, \delta) \ge \delta^{-\alpha_1}) \le \delta^{\alpha_1} E(N(\delta, K)) \le \delta^{\alpha_1 - \alpha_2}$

Now take $\delta_n = 2^{-n}$, observe that

$$\overline{\lim} \frac{\log E(N(\delta, K))}{|\log \delta|} = \overline{\lim_{n \to \infty}} \frac{\log E(N(2^{-n}, K))}{n \log 2}, \text{ and}$$

notice that $\sum P(N(K, 2^{-n}) \ge 2^{-n\alpha_1}) \le \sum 2^{n(\alpha_2 - \alpha_1)} < \infty$

So, by Borel-Cantelli, $N(K, 2^{-n}) \le 2^{-n\alpha_1}$ for large n

Proof that

$$\lim_{\delta \to 0} \frac{\log E(N(\delta, \gamma(0,\infty] \cap A))}{\log \frac{1}{\delta}} = 1 + \frac{\kappa}{8},\ \kappa \le 8.$$

for any open bounded A with $\overline{A} \subset \mathbb{H}$

Take $\delta < \mathrm{dist}(A, \mathbb{R})$.

Let $\mathcal{M} = \{ z \in \delta \mathbb{Z}^2 : \mathrm{dist}(z, A) \le \delta \}$. $m := \# \mathcal{M} \approx \frac{1}{\delta^2}$.

Observe $N(\delta, \gamma(0,\infty)\cap A) \le \#\{z\in \mathcal{M} : \mathrm{dist}(z,\gamma(0,\infty)) < \delta\}$

and $\#\{z\in \mathcal{M} : \mathrm{dist}(z,\gamma(0,\infty)) < \delta\} \le N(\delta, \gamma(0,\infty)\cap A)$

But $E(\#\{z\in \mathcal{M} : \mathrm{dist}(z,\gamma(0,\infty)) < \delta\}) = \sum_{z\in \mathcal{M}} P(\mathrm{dist}(z, \gamma(0,\infty)) < \delta)$

By scaling and boundedness of A,

$$P(\mathrm{dist}(z,\gamma(0,\infty)) < \delta) \asymp \sup_{|x|\le M} P(\mathrm{dist}(x+i, \gamma(0,\infty)) < C\delta)$$

for some M, C.

So, to establish the bound, we need $\exists c$.

$$c^{-1}\delta^{1-\frac{\kappa}{8}} \le P(\mathrm{dist}(x+i, \gamma(0,\infty)) < \delta) \le c\,\delta^{1-\kappa/8}, \quad |x|\le M. \qquad (*)$$

Reminder. $\mathrm{dist}(x+i, \gamma(0,\infty)) \asymp \exp(-D(x))$.

$$D(x) = \lim_{t\to\infty} \ln \frac{|h_t'(x+i)|}{\mathrm{Im}\, h_t(x+i)}$$

$$(*) \qquad E(e^{i b D(x)}) = \frac{\Gamma\left(\frac{2}{\kappa} + \sqrt{(\frac{2}{\kappa}-\frac{1}{2})^2 - i\frac{2b}{\kappa}}\right)\Gamma\left(\frac{2}{\kappa} - \sqrt{(\frac{2}{\kappa}-\frac{1}{2})^2 - i\frac{2b}{\kappa}}\right)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{4}{\kappa}-\frac{1}{2})} F(\alpha_+, \alpha_-, \tfrac{1}{2}, x).$$

We'll need the standard estimate:

Lemma. X – random variable, $\varphi(b) = E(\exp(ibX))$ – Characteristic function

Assume that for some $u, \lambda, \varepsilon > 0$:

$$\varphi(b) = \frac{u\lambda}{\lambda - ib} + V(b),$$

V – analytic on $\{|z| < \lambda + 2\varepsilon\}$.

Then $$P(X\ge x) = u e^{-\lambda x} + o(e^{-(\lambda+\varepsilon)x}) \quad (x\to\infty)$$

Proof. Let μ be distribution of X.

Let $\mu_1 = u\lambda e^{-\lambda x} 1_{x>0}\, dx$

$$\left(\text{and } \int e^{i\lambda b} d\mu_1(b) = \frac{u\lambda}{\lambda - ib}\right).$$

$\mu_2 = \mu - \mu_1$. Then $\int e^{i\lambda b} d\mu_2(b) = V(\lambda)$.

Since V is analytic in $\{|z| < \lambda + 2\varepsilon\}$,

$|\mu_2[x,\infty)| = o(e^{-(\lambda+\varepsilon)x})$ – inverse Fourier! So

$\mu_2([x,\infty)) = o(e^{-(\lambda+\varepsilon)x})$ — inverse Fourier! So

$$P(X \geq x) = \mu_1([x,\infty)) + \mu_2([x,\infty)) = ue^{-\lambda x} + o(e^{-(\lambda+\varepsilon)x})$$

From $(*)$, we can use Lemma for $\lambda = 1 - \frac{\kappa}{8}$,

$\varepsilon = \frac{2}{\kappa} + \frac{\kappa}{8} - 1.$

Indeed, $\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{z+k} + \int_1^{\infty} e^{-t} t^{z-1} dt$

$$\Gamma\left(\frac{2}{\kappa} - z\right)\Gamma\left(\frac{2}{\kappa} + z\right) = \frac{4/\kappa}{\frac{4}{\kappa^2} - z^2} + H(z),$$

where $H(z)$ is analytic in $\{|z| < \frac{2}{\kappa} + 1 = 1 + \varepsilon\}$

Plug in $(*)$